

Phenomenological Lagrangians, gauge models and branes

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Abstract

Phenomenological Lagrangians for physical systems with spontaneously broken symmetries are reformulated in terms of gauge field theory. Description of the Dirac p -branes in terms of the Yang-Mills-Cartan gauge multiplets interacting with gravity, is proved to be equivalent to their description as a closed dynamical system with the symmetry $ISO(1, D-1)$ spontaneously broken to $ISO(1, p) \times SO(D-p-1)$.

The geometric approach introduced for the description of string [1-5] considers its world-sheet as a surface embedded into 4-dim. Minkowski space $\mathbf{R}^{1,3}$. The gauge reformulation [6], [7] of the geometric approach represents the action of strings and p -branes in terms of the interacting Yang-Mills-Cartan multiplets and gravitational field localized on a $(p+1)$ -dim. world hypersurface Σ_{p+1} swept in $\mathbf{R}^{1,D-1}$. Using the Cartan formalism of moving frames [1] and its physical development [8-10] we interpret the above gauge description in terms of the Nambu-Goldstone (N-G) fields of the spontaneously broken Poincare symmetry $ISO(1, D-1)$ studied in [11-16]. Then string and p -brane emerge as the general solutions of the Euler-Lagrange EOM selected by the Maurer-Cartan eqs. that play the role of the Cauchy-Kovalevskaya initial data. We show that the generalized eqs. of the *Gauss Theorema Egregium* are the dynamical eqs. for the string and brane metrics. For the string embedded into $\mathbf{R}^{1,2}$ these eqs. represent the gravity described by the geometry of 2-dim. Einstein space.

1. Consider a global semisimple group of symmetry G spontaneously broken to H

$$[Y_\alpha, Y_\beta] = ic^\gamma_{\alpha\beta} Y_\gamma, \quad [X_i, Y_\alpha] = ic^k_{i\alpha} X_k, \quad [X_i, X_k] = ic^\alpha_{ik} Y_\alpha + ic^l_{ik} X_l. \quad (1)$$

and use the following factorized representation of its group elements [8], [18-21]

$$G(a, b) = K(a)H(b), \quad (2)$$

where a and b parametrize the group space of G . The left multiplication

$$gG = G' \quad \rightarrow \quad gK(a)H(b) = K(a')H(b') \quad (3)$$

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yields the nonlinear transformation of the group parameters

$$a' = a'(a, g), \quad b' = b'(b, a, g) \quad (4)$$

which preserves the differential form $G^{-1}dG$ and its components

$$G'^{-1}dG' = G^{-1}dG = i\omega_G^i(a, b, da)X_i + i\theta_G^\alpha(a, b, da, db)Y_\alpha. \quad (5)$$

The parameters a may be mapped into the components of the N-G field $\pi(x)$ with the same nonlinear transformation law $\pi'(x) = a'(\pi(x), g)$ for the construction of G -invariant nonlinear phenomenological Lagrangians for $\pi(x)$.

An alternative approach proposed for the chiral sigma models in [17], and generalized to strings in [6], is based on consideration of G as completely spontaneously broken symmetry when all group parameters (a, b) are treated as N-G modes. This extension is accompanied with a *compensating* extension of the *left global* symmetry of Lagrangian by a new *right gauge* symmetry H_R . The latter is realized by the *right* multiplications $G' = Gh$ with $h \in H_R$. In view of this gauge invariance, the non-physical b -modes associated with the parameters of the vacuum subgroup H are excluded by a gauge fixing. The transformation rules for G -invariant Cartan forms (5) under the right transformations from H_R are

$$G' = Gh \quad \rightarrow \quad G'^{-1}dG' = h^{-1}(G^{-1}dG)h + h^{-1}dh. \quad (6)$$

The substitution of expansion (5) into (6) using (1) gives the following gauge transformations

$$\omega_G'^i X_i = \omega_G^i h^{-1} X_i h, \quad \theta_G'^\alpha Y_\alpha = \theta_G^\alpha h^{-1} Y_\alpha h - i h^{-1} dh \quad (7)$$

for the left invariant one-forms ω_G^i and θ_G^α . This yields the transformation law

$$\delta\omega_G^k = -c^k_{i\beta}\epsilon^\beta\omega_G^i, \quad \delta\theta_G^\gamma = d\epsilon^\gamma - c^\gamma_{\alpha\beta}\epsilon^\beta\theta_G^\alpha \quad (8)$$

for the infinitesimal transformations $h \approx 1 + i\epsilon^\beta Y_\beta$ with the field-dependent parameters ϵ^β . This shows that these forms can be treated as one-forms of the massless vector and gauge multiplets of H_R built from the N-G fields. In the case of Dirac p -branes embedded into the Minkowski space $\mathbf{R}^{1,D-1}$ its global Lorentz symmetry $SO(1, D-1)$ plays the role of the discussed internal symmetry G , i.e. $G = G_{Lor} \equiv SO(1, D-1)$. The Lorentz symmetry is spontaneously broken to $SO(1, p) \times SO(D-p-1)$ due to the presence of brane. Thus, the rotational N-G fields can be described in terms of the left invariant one-forms (7). On the other hand, the rotational DOF of branes can be presented by the vectors $\mathbf{n}_A(\mathbf{x})$ of the Cartan moving frame in $\mathbf{R}^{1,D-1}$, where $\mathbf{x} = \{x^m\}$, $(m = 0, 1, \dots, D-1)$ are the global Cartesian coordinates. It shows that $\mathbf{n}_A(\mathbf{x})$ can encode the rotational N-G fields.

2. The moving frame in $\mathbf{R}^{1,D-1}$ is formed by the orthonormal vectors $\mathbf{n}_A(\mathbf{x})$ [1],

$$\begin{aligned} \mathbf{n}_A(\mathbf{x})\mathbf{n}_B(\mathbf{x}) &= \eta_{AB}, \quad (A, B = 0, 1, \dots, D-1), \\ d\mathbf{n}_A &= -\omega_A^B(d)\mathbf{n}_B, \quad d\mathbf{x} = \omega^A(d)\mathbf{n}_A \end{aligned} \quad (9)$$

with their vertex at the point \mathbf{x} . So, the frame is defined as the pair $(\mathbf{x}, \mathbf{n}_A(\mathbf{x}))$ called the moving D -hedron [22], [23]. In view of the $\frac{1}{2}D(D+1)$ constraints D^2 components n_{mA} of \mathbf{n}_A represent the pseudoorthogonal matrices $\hat{n} = n_{mA}$ parametrized by $\frac{1}{2}D(D-1)$ independent fields $\pi^\lambda(\mathbf{x})$. So, $\mathbf{n}_A = \mathbf{n}_A(\pi^\lambda)$, and π^λ can be treated as N-G bosons of the completely spontaneously broken $SO(1, D-1)$ symmetry realized by the left global multiplications

$$n'_{mA} = l_m^k n_{kA}, \quad l_m^p l_p^n = \delta_m^n, \quad (10)$$

where the matrices $\hat{l} \in SO(1, D-1)$. As an example, choose $SO_R(1, D-1)$ as such a gauge group acting to the right index A of \mathbf{n}_A

$$\mathbf{n}'_A = -L_A{}^B(\pi^\lambda)\mathbf{n}_B, \quad L_A{}^B L^C{}_B = \delta_A^C. \quad (11)$$

As a result, the matrix n_{mA} becomes double covariant under the left and right shifts

$$n'_{mA} = l_m{}^k n_{kB} L^B{}_A(\pi^\lambda). \quad (12)$$

But, in this case $SO_R(1, D-1)$ removes all the rotational N-G fields. It is instructive to understand this result in terms of the left form (5) invariant under the global Lorentz group

$$\omega_A{}^B(d) = (\hat{n}^{-1} d\hat{n})_A{}^B \equiv \mathbf{n}_A d\mathbf{n}^B. \quad (13)$$

Infinitesimal transformtaions of $\omega_A{}^B(d)$ under the right local rotations of $SO_R(1, D-1)$

$$\delta \mathbf{n}_A = \mathbf{n}_B \epsilon^B{}_A \quad (14)$$

turn out to be similar to the gauge transformations of a gauge field potential

$$\delta \omega_A{}^B(d) = d\epsilon_A{}^B + [\omega(d), \epsilon]_A{}^B \quad (15)$$

with the commutator $[\hat{\omega}, \hat{\epsilon}]$ in the r.h.s. of the law. The covariant differential $D_A{}^B$

$$D_A{}^B = d\delta_A{}^B + \omega_A{}^B(d). \quad (16)$$

for a vector field $\mathbf{V} = V^A \mathbf{n}_A$, transforming as $\delta V_A = V_B \epsilon^B{}_A$, has the homogeneous law

$$\delta(DV)_A = (DV)_B \epsilon^B{}_A. \quad (17)$$

The exterior product of differentials (16) yields the gauge covariant 2-form $F_A{}^B(d)$

$$F_A{}^B := D_A{}^C \wedge D_C{}^B = (d \wedge \omega + \omega \wedge \omega)_A{}^B. \quad (18)$$

The description of the N-G fields in terms of the gauge field $\hat{\omega}$ requires the integrability of Eqs. (13) that can be rewritten in the form of the PDEs

$$d\mathbf{n}_A = -\omega_A{}^B(d)\mathbf{n}_B. \quad (19)$$

The integrability conditions for the system (19) have the form

$$d \wedge \omega_A{}^B + \omega_A{}^C \wedge \omega_C{}^B = 0 \quad \rightarrow \quad F_A{}^B = 0. \quad (20)$$

So, we see that in the absence of the N-G fields the corresponding potential $\hat{\omega}$ must be a pure gauge form. This illustrates the equality of the DOF carried by the N-G fields and the gauge field $\omega_A{}^B$. When the Lorentz symmetry is partially broken to its vacuum subgroup $H \in SO(1, D-1)$, the right gauge subgroup $H_R \in SO(1, D-1)_R$ must be used instead of $SO(1, D-1)_R$. Then H_R will remove only the N-G fields corresponding to the generators of the subgroup H . That explains how one can change the standard description of the N-G fields by the transition to the vector and gauge multiplets.

For p -brane with the world vector $\mathbf{x}(\xi) = \{x^\mu(\xi)\}$, $(\xi^\mu = (\tau, \sigma^r), r = 1, 2, \dots, p)$ of its world hypersurface Σ_{p+1} , the vacuum subgroup $H = SO(1, p) \times SO(D-p-1)$. That forces to

choose $H_R = (SO(1, p) \times SO(D - p - 1))_R$ as the right gauge group. The latter is composed from the tangent Lorentz rotations accompanied with the rotations in $(D - p - 1)$ -dim. subspace normal to the plane tangent to Σ_{p+1} at a point $P(\mathbf{x}(\xi))$. That implies splitting of the local frame into the two subsets: $\mathbf{n}_A = (\mathbf{n}_i, \mathbf{n}_a)$, where \mathbf{n}_i , $(i, k = 0, 1, \dots, p)$ are tangent and \mathbf{n}_a $(a, b = p + 1, p + 2, \dots, D - p - 1)$ are normal to Σ_{p+1} at $P(\mathbf{x}(\xi))$. As a result, the Cartan form (13) for the global Lorentz group $SO(1, D - 1)$ is presented in the block form

$$\omega_A{}^B(d) = \begin{pmatrix} A_i{}^k(d) & W_i{}^b(d) \\ W_a{}^k(d) & B_a{}^b(d) \end{pmatrix}, \quad (21)$$

where $(p + 1)(D - p - 1)$ remaining N-G bosons are encoded in the bi-fundamental form $W_i{}^b(d)$ playing the role of ω_G^i in (5). The diagonal submatrices $A_{\mu i}{}^k d\xi^\mu$ and $B_{\mu a}{}^b d\xi^\mu$ in (21) describe the gauge forms in the fundamental reps. of $SO(1, p)$ and $SO(D - p - 1)$ subgroups corresponding to θ_G^α in (5), respectively. Then the integrability conditions (20) take the form

$$F_{\mu\nu i}{}^k = -(W_{[\mu} W_{\nu]})_i{}^k, \quad (22)$$

$$H_{\mu\nu a}{}^b = -(W_{[\mu} W_{\nu]})_a{}^b, \quad (23)$$

$$(D_{[\mu} W_{\nu]})_i{}^a = 0 \quad (24)$$

which yield the Gauss-Ricci-Codazzi (G-R-C) equations reformulated in terms of the massless vector multiplet $W_{\mu i}{}^a$ and the gauge strenghts $\hat{F}_{\mu\nu}$, $\hat{H}_{\mu\nu}$ in the curved space Σ_{p+1} with the induced metric $g_{\mu\nu}(\xi) = \partial_\mu \mathbf{x} \partial_\nu \mathbf{x}$. Invariance of Σ_{p+1} under diffeomorphisms requires extension of the gauge-covariant derivative $(D_\mu W_\nu)_i{}^a$ by the Levi-Chivita connection $\Gamma_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho$

$$(D_\mu W_\nu)_i{}^a \rightarrow (\hat{\nabla}_\mu W_\nu)_i{}^a = \partial_\mu W_{\nu i}{}^a + A_{\mu i}{}^k W_{\nu k}{}^a + B_{\mu a}{}^b W_{\nu i}{}^b - \Gamma_{\mu\nu}^\rho W_{\rho i}{}^a. \quad (25)$$

This covariantization does not change Eqs. (22-24), but adds the Riemann-Cristoffel tensor

$$[\hat{\nabla}_\mu, \hat{\nabla}_\nu] = \hat{F}_{\mu\nu} + \hat{H}_{\mu\nu} + \hat{R}_{\mu\nu} \quad (26)$$

in the commutator of the derivatives. The translational N-G fields induced by brane invariance under the Poincare group $ISO(1, D - 1)$ are represented by the projections of $d\mathbf{x}(\xi)$

$$\omega^A(d) = d\mathbf{x}(\xi) \mathbf{n}^A(\xi) \quad (27)$$

creating the form $\omega^A(d)$ (9) of the D -hedron. Encoding of the translational N-G modes $\mathbf{x}(\xi)$ by the forms $\omega^A(d)$ and $\mathbf{n}^A(\xi)$ is provided by the integrability conditions for PDEs (27)

$$d \wedge \omega_A + \omega_A{}^B \wedge \omega_B = 0. \quad (28)$$

The orthogonality conditions $\mathbf{n}_a(\xi) d\mathbf{x}(\xi) = 0$ result in the invariant constraints

$$\omega^a(d) = 0 \rightarrow d\mathbf{x} = \omega^i(d) \mathbf{n}_i(\xi) \quad (29)$$

which show that the quadratic element $ds^2 = d\mathbf{x}^2$ and $g_{\mu\nu}$ of Σ_{p+1} are presented in the form

$$ds^2 = \omega_i \omega^i = \omega_\mu^i \omega_{\nu i} d\xi^\mu d\xi^\nu \equiv g_{\mu\nu}(\xi) d\xi^\mu d\xi^\nu. \quad (30)$$

Substitution of the condition $\omega^a = 0$ into the M-C eqs. (28) results in the conditions

$$W_{\mu i}^a = -l_{\mu\nu}^a \omega_i^\nu, \quad l_{\mu\nu}^a := \mathbf{n}^a \partial_{\mu\nu} \mathbf{x}, \quad (31)$$

$$A_\mu^{ik} = \omega_\rho^i \Gamma_{\mu\lambda}^\rho \omega^{\lambda k} + \omega_\lambda^i \partial_\mu \omega^{\lambda k}, \quad F_{\mu\nu}^i{}_k = \omega_\gamma^i R_{\mu\nu}{}^\gamma{}_\lambda \omega_k^\lambda, \quad (32)$$

where $l_{\mu\nu}^a$ is the second fundamental form of Σ_{p+1} . Eqs. (32) show that the metric connection is equivalent to the gauge field A_μ^{ik} . Then invariance under $SO(1, p)$ is equivalent to that under diffeomorphisms of Σ_{p+1} . As a result, Eqs. (22-24) are transformed into

$$R_{\mu\nu}{}^\gamma{}_\lambda = l_{[\mu}{}^{\gamma a} l_{\nu]\lambda a}, \quad (33)$$

$$H_{\mu\nu}{}^{ab} = l_{[\mu}{}^{\gamma a} l_{\nu]\gamma}{}^b, \quad (34)$$

$$\nabla_{[\mu}^\perp l_{\nu]\rho a} = 0, \quad (35)$$

where the general and $SO(D - p - 1)$ covariant derivative ∇_μ^\perp is defined as

$$\nabla_\mu^\perp l_{\nu\rho}^a := \partial_\mu l_{\nu\rho}^a - \Gamma_{\mu\nu}^\lambda l_{\lambda\rho}^a - \Gamma_{\mu\rho}^\lambda l_{\nu\lambda}^a + B_\mu^{ab} l_{\nu\rho b}. \quad (36)$$

The commutator of two covariant derivatives ∇_μ^\perp (36) yields the Bianchi identities

$$[\nabla_\gamma^\perp, \nabla_\nu^\perp] l^{\mu\rho a} = R_{\gamma\nu}{}^\mu{}_\lambda l^{\lambda\rho a} + R_{\gamma\nu}{}^\rho{}_\lambda l^{\mu\lambda a} + H_{\gamma\nu}{}^a{}_b l^{\mu\rho b}. \quad (37)$$

Thus, we show that the rotational and translational N-G fields are encoded by $l_{\nu\rho}^a$ and ω_μ^i . The corresponding $SO(d - p - 1)$ gauge invariant phenomenological action of the Dirac p -brane in the long-wave approximation, selected by Eqs. (33-35) has the form [7]

$$S_{Dir.} = \gamma \int d^{p+1} \xi \sqrt{|g|} \left\{ -\frac{1}{4} Sp(H_{\mu\nu} H^{\nu\mu}) + \frac{1}{2} \nabla_\mu^\perp l_{\nu\rho a} \nabla^\perp \{^{\mu} l^{\nu\} \}^{\rho a} - \nabla_\mu^\perp l_{\rho a}^\mu \nabla_\nu^\perp l^{\nu\rho a} + V_{Dir.} \right\}, \quad (38)$$

$$V_{Dir.} = -\frac{1}{2} Sp(l_a l_b) Sp(l^a l^b) + Sp(l_a l_b l^a l^b) - Sp(l_a l^a l_b l^b) + c.$$

The action describes the interacting gauge B_μ^{ab} and tensor $l_{\mu\nu}^a$ fields in the background $g_{\mu\nu}$. Under construction of the potential term $V_{Dir.}$ Eqs. (33-35) have been taken into account together with the minimality condition $Sp l^a \equiv g^{\mu\nu} l_{\mu\nu}^a = 0$ invariant under all the symmetries of $S_{Dir.}$. The minimality conditions may be qualified as an inverse Higgs condition [24] similarly to the conditions $\omega^a(d) = 0$ (29) fixing the vacuum manifold for branes.

3. The Euler-Lagrange EOM for S_{Dirac} (38) are equivalently represented in the form of the second-order PDEs

$$\nabla_\nu^\perp \mathcal{H}_{ab}^{\nu\mu} = \frac{1}{2} l_{\nu\rho[a} \nabla^\perp [^\mu l^{\nu]}{}^{\rho}{}_{b}], \quad \mathcal{H}_{\mu\nu}^{ab} := H_{\mu\nu}^{ab} - l_{[\mu}{}^{\gamma a} l_{\nu]\gamma}{}^b, \quad (39)$$

$$\nabla_\mu^\perp \nabla^\perp [^\mu l^{\nu]}{}^{\rho a} = 0, \quad (40)$$

where $\mathcal{H}_{\mu\nu}^{ab}$ is the shifted strenght $H_{\mu\nu}^{ab}$ which presents Eqs. (34) and (35) as

$$\mathcal{H}_{\mu\nu}^{ab}(\tau, \sigma^r) = 0, \quad \nabla_{[\mu}^\perp l_{\nu]\rho a}(\tau, \sigma^r) = 0. \quad (41)$$

Noting that Eqs. (41) are the first order PDEs we will prove that they can be interpreted as the Cauchy initial data for PDEs (39) and (40). For this purpose let us consider Eqs. (41) as some constraints chosen at the time $\tau = 0$:

$$\mathcal{H}_{ab}^{\nu\mu}(0, \sigma^r) = 0, \quad \nabla^\perp [^\mu l^{\nu]}{}^{\rho a}(0, \sigma^r) = 0 \quad (42)$$

and analyze their time evolution prescribed by EOM (39), (40) considering the expansion

$$\begin{aligned}\mathcal{H}_{ab}^{\tau r}(\delta\tau, \sigma^r) &= \mathcal{H}_{ab}^{\tau r}(0, \sigma^r) + \partial_\tau \mathcal{H}_{ab}^{\tau r}|_{\tau=0} \delta\tau + \dots, \\ \nabla^{\perp[\tau} l^{r]\rho a}(\delta\tau, \sigma^r) &= \nabla^{\perp[\tau} l^{r]\rho a}(0, \sigma^r) + \partial_\tau \nabla^{\perp[\tau} l^{r]\rho a}|_{\tau=0} \delta\tau + \dots\end{aligned}\quad (43)$$

Using EOM (39), (40) and the initial data constraints (42) we transform Eqs.(43) as follows

$$\begin{aligned}\mathcal{H}_{ab}^{\tau r}(\delta\tau, \sigma^r) &= -\nabla_{r'}^{\perp} \mathcal{H}_{ab}^{r'r}|_{\tau=0} \delta\tau + \dots, \\ \nabla^{\perp[\tau} l^{r]\rho a}(\delta\tau, \sigma^r) &= -\nabla_{r'}^{\perp} \nabla^{\perp[r'} l^{r]\rho a}|_{\tau=0} \delta\tau + \dots\end{aligned}\quad (44)$$

Observing that the space covariant derivatives of (42) are equal to zero

$$\nabla_{r'}^{\perp} \mathcal{H}_{ab}^{\nu\mu}(0, \sigma^r) = \nabla_{r'}^{\perp} \nabla^{\perp[\mu} l^{\nu]\rho a}(0, \sigma^r) = 0, \quad (r' = 1, 2, \dots, p-1) \quad (45)$$

we find that Eqs. (41) are conserved in time in view of the dynamics given by (39), (40), the first pair of the extended Maxwell equations and the Ricci identities

$$\mathcal{H}_{\mu\nu}^{ab}(\delta\tau, \sigma^r) = \mathcal{H}_{\mu\nu}^{ab}(0, \sigma^r), \quad \nabla_{[\mu}^{\perp} l_{\nu]\rho a}(\delta\tau, \sigma^r) = \nabla_{[\mu}^{\perp} l_{\nu]\rho a}(0, \sigma^r). \quad (46)$$

Using the Cauchy-Kowalevskaya theorem of local existence and iniqueness we conclude that Eqs. (41) present the generic solution of EOM (39-40) corresponding to the initial data (42). The solution selects the closed sector of evolution states lying on the surface of the constraints (41) under an arbitrary but fixed metric $g_{\mu\nu}$. In its turn, the metric dynamics is defined by the Gauss condition (33) treated as the second-order PDEs for $g_{\mu\nu}$ with the given $l_{\mu\nu}^a$. In its turn the dynamics of the N-G multiplet $l_{\mu\nu}^a$ is derived using the standard variational principle for the action (38). For the Nambu-Goto string ($p=1$) in 3-dimensional Minkowski space the metric condition (33) is the Lioville equation proving that the string world-sheet is 2-dim. Einstein space with $R_{\mu\nu} = \kappa g_{\mu\nu}$ as we will show in the next section.

So, we prove that the action (38) describes the Dirac p -brane dynamics encoded in the quartic potential $V_{Dir.}$ selected by the G-R-C constraints (33-35) with $Spl^a = 0$. Note that a flat brane with $g_{\mu\nu} = \eta_{\mu\nu}$ corresponds to the evident solution $l_{\mu\nu}^a = 0$ and $V_{Dir.} = const.$

4. Here we illustrate the work of the above-discussed approach for a simple case of string ($p = 1$) embedded in 3-dim. Minkowski space studied in [6] (see also [5]). Using the gauge $l_{\tau\tau} = l_{\sigma\sigma} = 0$ and the minimality condition $Sp\ l=0$ (where $l_{\mu\nu} \equiv l_{\mu\nu 2}$) we find

$$l_{\mu\nu} = \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}, \quad Sp\ l = 0 \rightarrow g_{\mu\nu} = \begin{pmatrix} g_\tau & 0 \\ 0 & g_\sigma \end{pmatrix}, \quad M \neq 0. \quad (47)$$

In any 2-dim. space the Riemann tensor has only one non-zero component $R_{\tau\sigma\tau\sigma}$. Then the Gauss equation (33) are reduced to $R_{\tau\sigma\tau\sigma} = (l_{\tau\sigma})^2 \equiv M^2$, and we obtain

$$R_{\mu\nu} = M^2 \begin{pmatrix} 1/g_\sigma & 0 \\ 0 & 1/g_\tau \end{pmatrix}, \quad R = 2 \frac{M^2}{g_\tau g_\sigma} = -2 \frac{det l_{\mu\nu}}{det g_{\mu\nu}} \quad (48)$$

for the Ricci tensor $R_{\nu\lambda} = g^{\mu\gamma} R_{\mu\nu\gamma\lambda}$ and the scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$ of the worldsheet. From Eqs. (47), (48) we obtain the equation for 2-dim. Einstein space geometry

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \quad (49)$$

defining dynamics of the worldsheet metric $g_{\mu\nu}$ in the reformulated Nambu-Dirac action (38). Eq. (34) is identically satisfied since $B_\mu^{ab} \equiv 0$. Finally, Eqs. (35) take the following form

$$\partial_{[\mu} l_{\nu]\rho} - \Gamma_{\rho[\mu}^\lambda l_{\nu]\lambda} = 0, \quad (50)$$

$$\Gamma_{\tau\beta}^\alpha = \frac{1}{2} \begin{pmatrix} \partial_\tau g_\tau / g_\tau & \partial_\sigma g_\tau / g_\tau \\ -\partial_\sigma g_\tau / g_\sigma & \partial_\tau g_\sigma / g_\sigma \end{pmatrix}, \quad \Gamma_{\sigma\beta}^\alpha = \frac{1}{2} \begin{pmatrix} \partial_\sigma g_\tau / g_\tau & -\partial_\tau g_\sigma / g_\tau \\ \partial_\tau g_\sigma / g_\sigma & \partial_\sigma g_\sigma / g_\sigma \end{pmatrix}, \quad (51)$$

where α and β number the rows and columns. Insertion of (51) into (50) reduces it to

$$\partial_\tau M/M - \frac{1}{2}(\partial_\tau g_\tau / g_\tau - \partial_\tau g_\sigma / g_\sigma) = 0, \quad \partial_\sigma M/M + \frac{1}{2}(\partial_\sigma g_\tau / g_\tau - \partial_\sigma g_\sigma / g_\sigma) = 0. \quad (52)$$

The integrability condition of (52) gives $g_\sigma = -e^{2[-\varphi(\tau)+\chi(\sigma)]} g_\tau$ and we get the general solution of (52) $M = \pm \tilde{k} e^{\varphi(\tau)+\chi(\sigma)}$, ($\tilde{k} = \text{constant}$). The presence of arbitrary "integration constants" $\varphi(\tau)$ and $\chi(\sigma)$ in the solution is a consequence of the gauge symmetry of Eqs. (52)

$$M' = e^{\alpha(\tau)+\beta(\sigma)} M, \quad (g_\sigma / g_\tau)' = e^{-2[(\alpha(\tau)+\delta(\tau))-(\beta(\sigma)+\gamma(\sigma))]} (g_\sigma / g_\tau), \quad (53)$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary functions, as follows from Eqs. (52) represented in the form: $\partial_\tau \ln(M^2 |g_\sigma / g_\tau|) = 0$, $\partial_\sigma \ln(M^2 |g_\tau / g_\sigma|) = 0$. So, one can choose the conformal gauge

$$g_\sigma = -g_\tau \equiv -E(\tau, \sigma), \quad M^2 = k, \quad R_{\tau\sigma\tau\sigma} = k \quad (54)$$

So, the rotational N-G field M is a solution of Eq. $\square M = 0$ for a massless scalar field in the 2-dim. conformal flat space-time. To solve the H-E Eq. (49) we use the definition

$$R_{\tau\sigma\tau\sigma} = E R_{\tau\sigma}{}^\tau{}_\sigma = E(\partial_{[\tau} \Gamma_{\sigma]\sigma}^\tau + \Gamma_{\rho[\tau}^\tau \Gamma_{\sigma]\sigma}^\rho) = \frac{1}{2E} [E \square E - (\partial_\tau E)^2 + (\partial_\sigma E)^2] \quad (55)$$

for the Riemann tensor and representation (51) referred to the conformal gauge. Making the change $E := e^{2\psi}$ and using Eq. (54) we transform Eq. (49) into the Liouville one

$$(\partial_\tau^2 - \partial_\sigma^2)\psi = k e^{-2\psi} \quad (56)$$

earlier proved to describe the relativistic string ($p = 1, D = 3$) in the geometrical approach [4]. This shows the equivalence of the gauge approach to the standard approach resulting in 2-dim. conformal invariant EOM (56) for the string metrics on the classical level.

Summary:

The Nambu-Goldstone fields of the spontaneously broken internal symmetries were described as the effective massless Yang-Mills-Cartan multiplets. The interpretation of the Dirac p -brane [7] embedded into $\mathbf{R}^{1,D-1}$ as a dynamical system with the $ISO(1, D-1)$ symmetry spontaneously broken to $ISO(1, p) \times SO(D-p-1)$ was established. The brane metric dynamics was shown to be described by the generalized eqs. of the *Gauss Theorema Egregium*. For the case of string embedded into $\mathbf{R}^{1,2}$ the geometry of its worldsheet turned out to coincide with the geometry of 2-dim. Einstein space. Having in mind a connection between the *quantum* conformal invariance in string theory, vanishing of its beta function and the vacuum H-E eqs. [25], [26] it seems interesting to find its generalization to branes.

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